

MAS321 Section 5.1 Exercise 4

Images of the Gauss Map

Group A: 김민영, 이종서, 이찬유, 정재웅, 최희준, 홍성제

KAIST

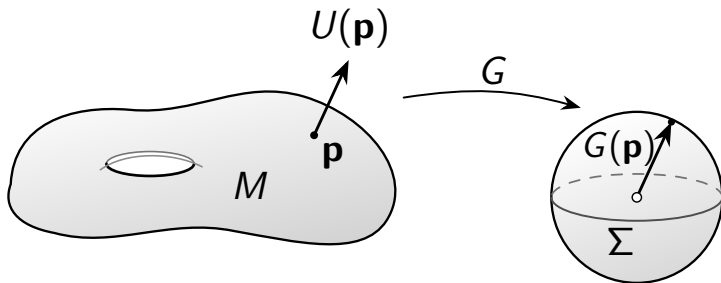
Problem 4

The Gauss map

Let surface $M \subset \mathbb{R}^3$ have a unit normal field $U = g_1 U_1 + g_2 U_2 + g_3 U_3$. The Gauss map is

$$G : M \longrightarrow \Sigma, \quad G(p) = (g_1(p), g_2(p), g_3(p)),$$

where unit sphere $\Sigma = \{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$.



Problem 4

The Gauss map

Let surface $M \subset \mathbb{R}^3$ have a unit normal field $U = g_1 U_1 + g_2 U_2 + g_3 U_3$. The Gauss map is

$$G : M \longrightarrow \Sigma, \quad G(p) = (g_1(p), g_2(p), g_3(p)),$$

where unit sphere $\Sigma = \{(X, Y, Z) : X^2 + Y^2 + Z^2 = 1\}$.

Describe $G(M)$ for each surface, using either normal:

- (a) Cylinder: $x^2 + y^2 = r^2, \quad r > 0$.
- (b) Cone: $z = \sqrt{x^2 + y^2}$.
- (c) Plane: $x + y + z = 0$.
- (d) Sphere: $(x - 1)^2 + y^2 + (z + 2)^2 = 1$.

Computing the Gauss map

A unit normal from a level surface

If $M = \{F = 0\}$ and $\nabla F \neq 0$ on M , choose

$$U(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}.$$

Its three Euclidean components give $G(p)$.

To describe the image, we will:

- 1 Compute the normalized normal direction.
- 2 Find the conditions satisfied by $(X, Y, Z) = G(p)$.
- 3 Check that every point in the proposed image occurs.

Geometric meaning: translate each unit normal to the origin and record its tip on the unit sphere.

Solution: 4-(a), cylinder

For $F = x^2 + y^2 - r^2$, choose the outward normal:

$$\nabla F = (2x, 2y, 0), \quad G(x, y, z) = \left(\frac{x}{r}, \frac{y}{r}, 0 \right).$$

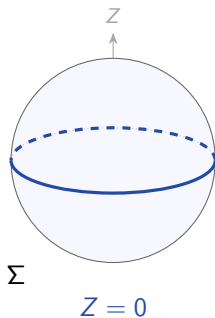
Write $p = (r \cos \theta, r \sin \theta, z)$. Then

$$G(p) = (\cos \theta, \sin \theta, 0).$$

As θ varies, every point of the equator occurs. The height z has no effect.

Image: a great circle

$$G(M) = \{(X, Y, Z) \in \Sigma : Z = 0\}.$$



Solution: 4-(b), cone

Work on the smooth part of the cone, with $\rho = \sqrt{x^2 + y^2} > 0$.
For $F = z - \sqrt{x^2 + y^2}$, choose the upward normal:

$$\nabla F = \left(-\frac{x}{\rho}, -\frac{y}{\rho}, 1 \right), \quad \|\nabla F\| = \sqrt{2}.$$

Writing $p = (\rho \cos \theta, \rho \sin \theta, \rho)$ gives

$$G(p) = \frac{1}{\sqrt{2}}(-\cos \theta, -\sin \theta, 1).$$

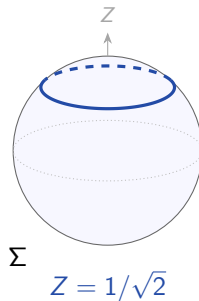
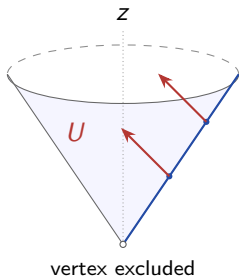
Every angle occurs, and ρ has no effect. Hence

Image: a small circle

$$G(M) = \left\{ (X, Y, Z) \in \Sigma : Z = \frac{1}{\sqrt{2}} \right\}.$$

Its center is $(0, 0, 1/\sqrt{2})$ and its radius is $1/\sqrt{2}$.

Solution: 4-(b), geometry of the cone



Along a fixed generator, θ is constant, so U is constant. Rotating around the cone traces the entire latitude circle.

Solution: 4-(c), plane

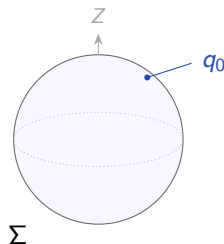
For $F = x + y + z$, the gradient is constant:

$$\nabla F = (1, 1, 1), \quad G(p) = q_0 := \frac{1}{\sqrt{3}}(1, 1, 1) \quad \text{for every } p \in M.$$

All normals have the same direction.
Thus every point of the plane maps to
one point of Σ .

Image: a single point

$$G(M) = \left\{ \frac{1}{\sqrt{3}}(1, 1, 1) \right\}.$$



Solution: 4-(d), translated unit sphere

The center is $a = (1, 0, -2)$, and $\|p - a\| = 1$ on M . The outward unit normal gives

$$G(p) = p - a = (x - 1, y, z + 2).$$

For any $q \in \Sigma$, take $p = a + q$. Then

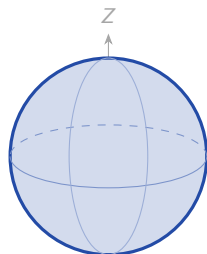
$$\|p - a\| = \|q\| = 1,$$

so $p \in M$ and $G(p) = q$.

Image: the entire unit sphere

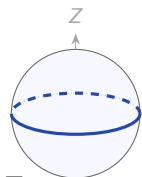
$$G(M) = \Sigma.$$

The translation of the center does not restrict the normal directions.

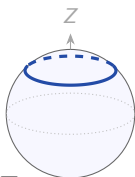
 Σ

all of Σ

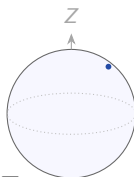
The four Gauss-map images



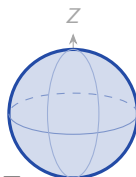
Σ
(a) Cylinder



Σ
(b) Cone



Σ
(c) Plane



Σ
(d) Sphere

Surface	Chosen normal	Image in Σ
Cylinder	Outward	Great circle: $Z = 0$
Cone	Upward	Small circle: $Z = 1/\sqrt{2}$
Plane	Toward $(1, 1, 1)$	Point: $(1, 1, 1)/\sqrt{3}$
Sphere	Outward	Entire sphere Σ

Changing U to $-U$ replaces the image by its antipodal set.